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A Functorial Bridge between the Infinitary Affine Lambda-Calculus and Linear Logic

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Abstract. It is a well known intuition that the exponential modality of linear logic may be seen as a form of limit. Recently, Melliès, Tabareau and Tasson gave a categorical account for this intuition, whereas the first author provided a topological account, based on an infinitary syntax. We relate these two different views by giving a categorical version of the topological construction, yielding two benefits: on the one hand, we obtain canonical models of the infinitary affine lambda-calculus introduced by the first author; on the other hand, we find an alternative formula for computing free commutative comonoids in models of linear logic with respect to the one presented by Melliès et al.

1 Introduction

The exponential modality of linear logic as a limit. Following the work of Girard [5], linearity has become a central notion in computer science and proof theory: it provides a finer-grained analysis of cut-elimination, which in turn, via Curry-Howard, gives finer tools for the analysis of the execution of programs. It is important to observe that the expressiveness of strictly linear or affine calculi is severely restricted, because programs in these calculi lack the ability to duplicate their arguments. The power of linear logic (which, in truth, is not linear at all!) resides in its so-called *exponential modalities*, which allow duplication (and erasing, if the logic is not already affine).

A possible approach to understand exponentials is to see the non-linear part of linear logic as a sort of limit of its purely linear part. The following old result morally says that, in the propositional case, exponential-free linear logic is "dense" in full linear logic:

Theorem 1 (Approximation [5]). *Define the* bounded exponential

$$!_p A := \overbrace{(A \mathbin{\&} 1) \otimes \cdots \otimes (A \mathbin{\&} 1)}^{p\ times},$$

and define $?_p A := (!_p A^\perp)^\perp$. Note that these formulas are exponential-free (if A is). Let A be a propositional formula with m occurrences of the $!$ modality *and n occurrences of the $?$ modality. If A is provable in full linear logic, then for every $p_1, \ldots, p_m \in \mathbf{N}$ there exist $q_1, \ldots, q_n \in \mathbf{N}$ such that A' is provable in exponential-free linear logic, where A' is obtained from A by replacing the i-th occurrence of $!$ with $!_{p_i}$ and the j-th occurrence of $?$ with $?_{q_j}$.*

For example, from the canonical proof of $?A^\perp \parr (!A \otimes !A)$ (contraction, *i.e.* duplication), we get proofs of $?_{p_1+p_2}A^\perp \parr (!_{p_1}A \otimes !_{p_2}A)$ for all $p_1, p_2 \in \mathbf{N}$.

Remember that, if a linear formula A says "A exactly once", then $!A$ stands for "A at will". The formula $A \,\&\, 1$ is an affine version of A: it says "A at most once". This is a very specialized use of additive conjunction, in the sequel we prefer to avoid additive connectives and denote the affine version of A by $A^\bullet$, which may or may not be defined as $A \,\&\, 1$ (for instance, in affine logic, $A^\bullet = A$). Therefore, $!_pA = (A^\bullet)^{\otimes p}$ stands for "A at most p times", hence the name bounded exponential. So the Approximation Theorem supports the idea that $!A$ is somehow equal to $\lim_{p\to\infty} !_pA$.

Categories vs. topology. This idea was recently formalized in two quite different ways. The first is due to Melliès, Tabareau and Tasson [14], who rephrased the question in categorical terms. It is well known [3] that a $*$-autonomous category admitting the free commutative comonoid A^∞ on every object A is a model of linear logic (a so-called *Lafont category*). So, given a Lafont category, how does one compute A^∞? Using previous work by the first two authors [13], Melliès et al. showed that one may proceed as follows:

- compute the free co-pointed object $A^\bullet$ on A (which is $A \,\&\, 1$ if the category has binary products);
- compute the symmetric versions of the tensorial powers of $A^\bullet$, *i.e.* the following equalizers, where $\mathfrak{S}_n$ is the set of canonical symmetries of $(A^\bullet)^{\otimes n}$:

$$A^{\leqslant n} \longrightarrow (A^\bullet)^{\otimes n} \circlearrowleft \mathfrak{S}_n$$

- compute the following projective limit, where $A^{\leqslant n} \longleftarrow A^{\leqslant n+1}$ is the canonical arrow "throwing away" one component:

$$\begin{tikzcd}
1 & A^{\leqslant 1} \arrow[l] & A^{\leqslant 2} \arrow[l] & \cdots \arrow[l] & A^{\leqslant n} \arrow[l] & \cdots \arrow[l] \\
 & & & A^\infty \arrow[ulll] \arrow[ull] \arrow[ul] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

At this point, for A^∞ to be the commutative comonoid on A it is enough that all relevant limits (the equalizers and the projective limit) commute with the tensor. Although not valid in general, this condition holds in several Lafont categories of very different flavor, such as Conway games and coherence spaces.

The second approach, due to the first author [10], is topological, and is based directly on the syntax. One considers an affine λ-calculus in which variables are treated as bounded exponentials: in a term of this calculus, a variable x may appear any number of times, each occurrence appears indexed by an integer (each instance, noted x_i, is labelled with a distinct $i \in \mathbf{N}$). The argument of applications is not a term but a sequence of terms, and to reduce the redex $(\lambda x.t)\langle u_0, \ldots, u_{n-1}\rangle$ one replaces each free x_i in t with u_i (a special term $\perp$ is substituted if $i \geq n$). The calculus is therefore affine, in the sense that no duplication is performed, and in fact it strongly normalizes even in absence of types (the size of terms strictly decreases with reduction).

At this point, the set of terms is equipped with the structure of uniform space[3], the Cauchy-completion of which, denoted by $\Lambda_\infty^{\mathrm{aff}}$, contains infinitary terms, *i.e.* allowing infinite sequences $\langle u_1, u_2, u_3, \ldots\rangle$. The original calculus embeds (and is dense) in $\Lambda_\infty^{\mathrm{aff}}$ by considering a finite sequence as an almost-everywhere $\perp$ sequence. Reduction, which is continuous, is defined as above, except that infinitely many substitutions may occur. This yields non-termination, in spite of the calculus still being affine: if $\Delta_n := \lambda x.x_0\langle x_1, \ldots, x_n\rangle$, then $\Delta := \lim_{n\to\infty} \Delta_n = \lambda x.x_0\langle x_1, x_2, x_3, \ldots\rangle$ and $\Omega := \Delta\langle \Delta, \Delta, \Delta, \ldots\rangle \to \Omega$.

Ideally, these infinitary terms should correspond to usual λ-terms. But there is a continuum of them, definitely too many. The solution is to consider a partial equivalence relation $\approx$ such that, in particular, $x_i \approx x_j$ for all i, j and $t\langle u_1, u_2, u_3, \ldots\rangle \approx t'\langle u'_1, u'_2, u'_3, \ldots\rangle$ whenever $t \approx t'$ and, for all $i, i' \in \mathbf{N}$, $u_i \approx u'_{i'}$. After introducing a suitable notion of reduction $\Rightarrow$ on the equivalence classes of $\approx$, one finally obtains the isomorphism for the reduction relations

$$(\Lambda_\infty^{\mathrm{aff}}/{\approx}, \Rightarrow) \quad \cong \quad (\Lambda, \to_\beta),$$

where $(\Lambda, \to_\beta)$ is the usual pure λ-calculus with β-reduction. Similar infinitary calculi (also with a notion of partial equivalence relation) were considered by Kfoury [7] and Melliès [11], although without a topological perspective. The indices identifying the occurrences of exponential variables are also reminiscent of Abramsky, Jagadeesan and Malacaria's games semantics [1].

Reconciling the two approaches. The contribution of this paper is to draw a bridge between the two approaches presented above. Indeed, we develop a categorical version of the topological construction of [10], which turns out to:

1. give a canonical way of building denotational models of the infinitary affine λ-calculus;
2. provide an alternative formula for computing the free commutative comonoid in a Lafont category.

Drawing inspiration from [13,14], we base our work on functorial semantics in the sense of Lawvere, computing free objects as Kan extensions.

Functorial semantics. The idea of functorial semantics is to describe an algebraic theory as a certain category constituted of the different powers of the domain of the theory as the objects, the operations of the theory as morphisms, and encode the relations between the operations in the composition operation. We will not consider algebraic theories as Lawvere did, but the more general symmetric monoidal theories, or PROPs [8] (*product and permutation categories*).

[3] The generalization of a metric space, still allowing one to speak of Cauchy sequences.

Definition 1 (symmetric monoidal theory). *An n-sorted* symmetric monoidal theory *is defined as a symmetric monoidal category $\mathbb{T}$ whose objects are n-tuples of natural numbers and with a tensorial product defined as the point-wise arithmetical sum.*

A model *of $\mathbb{T}$ in a symmetric monoidal category (SMC) $\mathcal{C}$ is a symmetric strong monoidal functor $\mathbb{T} \to \mathcal{C}$.*

A morphism *of models of $\mathbb{T}$ in $\mathcal{C}$ is a monoidal natural transformation between models of $\mathbb{T}$ in $\mathcal{C}$. We will denote as $\mathrm{Mod}(\mathbb{T}, \mathcal{C})$ the category with models of $\mathbb{T}$ in $\mathcal{C}$ as objects and morphisms between models as morphisms.*

The simplest symmetric monoidal theory, denoted by $\mathbb{B}$, has as objects the natural numbers seen as finite ordinals and as morphisms the bijections between them (the permutations). Alternatively, $\mathbb{B}$ can be seen as the free symmetric monoidal category on one object (the object 1, with monoidal unit 0). As such, a model of $\mathbb{B}$ is nothing but an object A in a symmetric monoidal category $\mathcal{C}$, and the categories $\mathcal{C}$ and $\mathrm{Mod}(\mathbb{B}, \mathcal{C})$ are equivalent.

The key non-trivial example in our context is that of commutative (co)monoids. We remind that a commutative monoid in a SMC $\mathcal{C}$ is a triple $(A, \mu : A \otimes A \to A, \eta : \mathbf{1} \to A)$, with A an object of $\mathcal{C}$, such that the arrows μ and η interact with the associator, unitors and symmetry of $\mathcal{C}$ to give the usual laws of associativity, neutrality and commutativity (see *e.g.* [9]). A morphism of monoids $f : (A, \mu, \eta) \to (A', \mu', \eta')$ is an arrow $f : A \to A'$ such that $f \circ \mu = \mu' \circ (f \otimes f)$ and $f \circ \eta = \eta'$. We denote the category of monoids of $\mathcal{C}$ and their morphisms as $\mathrm{Mon}(\mathcal{C})$. The dual notion of comonoid, and the relative category $\mathrm{Comon}(\mathcal{C})$, is obtained by reversing the arrows in the above definition. Now, consider the symmetric monoidal theory $\mathbb{F}$ whose objects are the natural numbers seen as finite ordinals and its morphisms are the functions between them (*i.e.* $\mathbb{F}$ is the skeleton of the category of finite sets). We easily check that $\mathrm{Mod}(\mathbb{F}, \mathcal{C}) \simeq \mathrm{Mon}(\mathcal{C})$ and $\mathrm{Mod}(\mathbb{F}^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Comon}(\mathcal{C})$. Indeed, a strict symmetric monoidal functor from $\mathbb{F}$ to $\mathcal{C}$ picks an object of $\mathcal{C}$ and the image of any arrow $m \to n$ of $\mathbb{F}$ is unambiguously obtained from the images of the unique morphisms $0 \to 1$ and $2 \to 1$ in $\mathbb{F}$, which are readily verified to satisfy the monoid laws.

Summing up, finding the free commutative comonoid A^∞ on an object A of a SMC $\mathcal{C}$ is the same thing as turning a strict symmetric monoidal functor $\mathbb{B} \to \mathcal{C}$ into a strict symmetric monoidal functor $\mathbb{F}^{\mathrm{op}} \to \mathcal{C}$ which is universal in a suitable sense. This is where Kan extensions come into the picture.

Free comonoids as Kan extensions. Kan extensions allow to extend a functor along another. Let $K : \mathcal{C} \to \mathcal{D}$ and $F : \mathcal{C} \to \mathcal{E}$ be two functors. If we think of K as an inclusion functor, it seems natural to try to define a functor $\mathcal{D} \to \mathcal{E}$ that would in a sense be universal among those that extend F. There are two ways of formulating this statement precisely, yielding left and right Kan extensions. We only describe the latter, because it is the case of interest for us:

Definition 2 (Kan extension). *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}$ be three categories and $F : \mathcal{C} \to \mathcal{E}$, $K : \mathcal{C} \to \mathcal{D}$ two functors. The* right Kan extension of F along K *is a functor $\mathrm{Ran}_K F : \mathcal{D} \to \mathcal{E}$ together with a natural transformation $\varepsilon : \mathrm{Ran}_K F \circ K \Rightarrow F$*

such that for any other pair $(G : \mathcal{D} \to \mathcal{E}, \gamma : G \circ K \Rightarrow F)$, γ factors uniquely through ε:

$$
\begin{tikzcd}
\mathcal{C} \arrow[rr, "K"] \arrow[dr, bend right, "F"'] & & \mathcal{D} \arrow[dl, dashed] \arrow[dl, bend left, "G"] \\
& \mathcal{E} \arrow[u, phantom, "\Leftarrow \varepsilon"] &
\end{tikzcd}
$$

It is easy to check that $\mathbf{Cat}(G, \mathrm{Ran}_K F) \simeq \mathbf{Cat}(G \circ K, F)$, where by $\mathbf{Cat}(f, g)$ (f and g being functors with same domain and codomain) we mean the 2-homset of the 2-category $\mathbf{Cat}$, *i.e.* the set of all natural transformations from f to g. In other words, Ran_K is right adjoint to U_K, the functor precomposing with K (whence the terminology "right"—the left adjoint to U_K is the left Kan extension). This observation is important because it tells us that Kan extensions may be relativized to any 2-category. In particular, we may speak of *symmetric monoidal Kan extensions* by taking the underlying 2-category to be **SymMonCat** (symmetric monoidal categories, strict symmetric monoidal functors and monoidal natural transformations).

Now, there is an obvious inclusion functor $i : \mathbb{B} \to \mathbb{F}^{\mathrm{op}}$ (bijections are particular functions), which is strictly symmetric monoidal. So if $\mathcal{E}$ is symmetric monoidal and A is an object of $\mathcal{E}$, we are in the situation described above with $\mathcal{C} = \mathbb{B}$, $\mathcal{D} = \mathbb{F}^{\mathrm{op}}$, $K = i$ and F the strict symmetric monoidal functor corresponding to A, which we abusively denote by A. The fundamental difference is that the diagram lives in **SymMonCat** instead of **Cat**. It is an instructive exercise to verify that the free commutative comonoid on A, if it exists, is $A^\infty = \mathrm{Ran}_i A(1)$, *i.e.* the right symmetric monoidal Kan extension of A along i, computed in 1:

$$
\begin{tikzcd}
\mathbb{B} \arrow[rr, "i"] \arrow[dr, bend right, "A"'] & & \mathbb{F}^{\mathrm{op}} \arrow[dl, bend left, "\mathrm{Ran}_i A"] \\
& \mathcal{E} \arrow[u, phantom, "\Leftarrow"] &
\end{tikzcd}
$$

Remember that the free commutative comonoid on A is a commutative comonoid A^∞ with an arrow $\mathrm{d} : A^\infty \to A$ such that, whenever C is a commutative comonoid and $f : C \to A$, there is a unique comonoid morphism $u : C \to A^\infty$ such that $f = \mathrm{d} \circ u$. The arrow d is ε, where $\varepsilon : \mathrm{Ran}_i A \circ i \Rightarrow A$ is the natural transformation coming with the Kan extension.

More generally, if $\mathbb{T}_1$ and $\mathbb{T}_2$ are two symmetric monoidal theories, a symmetric monoidal functor $i : \mathbb{T}_1 \to \mathbb{T}_2$ induces a forgetful functor $U_i : \mathrm{Mod}(\mathbb{T}_2, \mathcal{E}) \to \mathrm{Mod}(\mathbb{T}_1, \mathcal{E})$ such that $M \mapsto M \circ i$. So we may reformulate the problem of finding the "free $\mathbb{T}$-model" on an object A of $\mathcal{E}$ as finding a left monoidal adjoint to U_i with $i : \mathbb{B} \to \mathbb{T}$. That is precisely what we did above, with $\mathbb{T} = \mathbb{F}^{\mathrm{op}}$.

Computing monoidal Kan extensions. The above discussion is interesting because it provides a way of explicitly computing A^∞ from A. In fact, there is a well-known formula for computing Kan extensions [9]. When applied to the above special case, it gives

$$A^{\infty} = \prod_n A^{\otimes n}/\!\sim,$$

where $A^{\otimes n}/\sim$ is the symmetric tensor product. However, this formula works only for Kan extensions in **Cat** and there are no known formulas in other 2-categories. The main contribution of [13] was to find a sufficient condition under which the formula is correct also in **SymMonCat**. The condition is, roughly speaking, a commutation of the tensor with certain limits depending on the Kan extension at stake. In the above case, it requires the tensor to commute with countable products, which, in models of linear logic, boils down to having countable biproducts. Lafont categories of this kind do exist (*e.g.* the category **Rel** of sets and relations), but they are a little degenerate and not very representative.

The idea of [14] was to decompose the Kan extension in two, so that the commutation condition is weaker and satisfied by more Lafont categories. The intermediate step uses a symmetric monoidal theory denoted by $\mathbb{I}$, whose objects are natural numbers (seen as finite ordinals) and morphisms are the injections. Note that $\mathrm{Mod}(\mathbb{I}^{\mathrm{op}}, \mathcal{C})$ is equivalent to the slice category $\mathcal{C} \downarrow \mathbf{1}$. By definition, this is the category of *copointed objects* of $\mathcal{C}$: pairs $(A, w : A \to \mathbf{1})$ (with $\mathbf{1}$ the tensor unit, not necessarily terminal), with morphisms $f : (A, w) \to (A', w')$ arrows $f : A \to A'$ such that $w = w' \circ f$[4].

There are of course strict symmetric monoidal injections $j : \mathbb{B} \to \mathbb{I}^{\mathrm{op}}$ and $j' : \mathbb{I}^{\mathrm{op}} \to \mathbb{F}^{\mathrm{op}}$, such that $j' \circ j = i$. Unsurprisingly, $\mathrm{Ran}_j A(1)$ is the free copointed object on A, which we denoted by $A^{\bullet}$ above. Since Kan extensions compose (assuming they exist), we have $A^{\infty} = \mathrm{Ran}_{j'} A^{\bullet}(1)$:

$$\begin{tikzcd}
\mathbb{B} \arrow[r, "j"] \arrow[dr, bend right, "A"'] & \mathbb{I}^{\mathrm{op}} \arrow[r, "j'"] \arrow[d, "A^{\bullet}"] \arrow[dl, phantom, "\Leftarrow"] & \mathbb{F}^{\mathrm{op}} \arrow[dl, bend left, "A^{\infty}"] \arrow[dl, phantom, "\Leftarrow"] \\
& \mathcal{C} &
\end{tikzcd}$$

For the second Kan extension to be computed in **SymMonCat** using the **Cat** formula, a milder commutation condition than requiring countable biproducts suffices. It is the commutation condition we mentioned above when we recalled the three-step computation of A^{∞} (free copointed object, equalizers, projective limit), which indeed results from specializing the general Kan extension formula.

One more intermediate step. The bridge between the categorical and the topological approach will be built upon a further decomposition of the Kan extension: in the second step, we interpose a 2-sorted theory, denoted by $\mathbb{P}$ (this is why we introduced multi-sorted theories, all theories used so far are 1-sorted):

[4] The w stands for weakening.

$$
\begin{tikzcd}
\mathbb{B} \arrow[r, "j"] \arrow[rrd, bend right, "A"'] & \mathbb{I}^{\mathrm{op}} \arrow[r, "k"] \arrow[rd, bend right, "A^\bullet"'] & \mathbb{P} \arrow[r, "l"] \arrow[d, "(A^\omega, A^\bullet)"'] & \mathbb{F}^{\mathrm{op}} \arrow[ld, bend left, "A^\infty"] \\
& & \mathcal{C} &
\end{tikzcd}
$$

We will call the models of $\mathbb{P}$ *partitionoids*. Intuitively, the free partitionoid on A allows to speak of infinite streams on $A^\bullet$, from which one may extract arbitrary elements and substreams via maps of type $A^\omega \to (A^\bullet)^{\otimes m} \otimes (A^\omega)^{\otimes n}$. Such maps are the key to model the infinitary affine λ-calculus. This intuition is especially evident in **Rel** (the category of sets and relations), where A^ω is the set of all functions $\mathbf{N} \to A^\bullet$ which are almost everywhere $*$ (in **Rel**, $A^\bullet = A \uplus \{*\}$).

2 The Infinitary Affine Lambda-Calculus

We consider three pairwise disjoint, countable sets of *linear*, *affine* and *exponential* variables, ranged over by $k, l, m \ldots$, $a, b, c \ldots$ and $x, y, z \ldots$, respectively. The terms of the infinitary affine λ-calculus belong to the following grammar:

$$
\begin{array}{lll}
t, u ::= l \mid \lambda l.t \mid tu \mid \mathsf{let}\, k \otimes l = u \,\mathsf{in}\, t \mid t \otimes u & & \text{linear} \\
\quad\quad\; \mid a \mid \mathsf{let}\, a^\bullet = u \,\mathsf{in}\, t \mid \bullet t & & \text{affine} \\
\quad\quad\; \mid x_i \mid \mathsf{let}\, x^\omega = u \,\mathsf{in}\, t \mid \langle u_0, u_1, u_2, \ldots \rangle & & \text{exponential}
\end{array}
$$

The linear part of the calculus comes from [2]. It is the internal language of symmetric monoidal closed categories. As usual, let constructs are binders. The notation $\langle u_0, u_1, u_2, \ldots \rangle$ stands for an infinite sequence of terms. We use $\mathbf{u}$ to range over such sequences and write $\mathbf{u}(i)$ for u_i. Note that each u_i is inductively smaller than $\mathbf{u}$, so terms are infinite but well-founded. The usual linearity/affinity constraints apply to linear/affine variables, with the additional constraint that if x_i, x_j are distinct occurrences of an exponential variable in a term, then $i \neq j$. Furthermore, the free variables of a term of the form $\mathbf{u}$ (resp. $\bullet t$) must all be exponential (resp. exponential or affine).

The reduction rules are as follows:

$$
\begin{array}{rr}
(\lambda l.t)u \to t[u/l] & \quad \mathsf{let}\, k \otimes l = u \otimes v \,\mathsf{in}\, t \to t[u/k][v/l] \\
\mathsf{let}\, a^\bullet = \bullet u \,\mathsf{in}\, t \to t[u/a] & \quad \mathsf{let}\, x^\omega = \mathbf{u} \,\mathsf{in}\, t \to t[\mathbf{u}(i)/x_i]
\end{array}
$$

In the exponential rule, i ranges over $\mathbf{N}$, so there may be infinitely many substitutions to be performed. There are also the usual commutative conversions involving let binders, which we omit for brevity. The reduction is confluent, as the rules never duplicate any subterm.

The results of [10] are formulated in an infinitary calculus with exponential variables only, whose terms and reduction are defined as follows:

$$
t, u ::= x_i \mid \lambda x.t \mid t\langle u_0, u_1, u_2, \ldots \rangle, \qquad\qquad (\lambda x.t)\mathbf{u} \to t[\mathbf{u}(i)/x_i]
$$

$$\frac{}{\Gamma; \Delta; l : A \vdash l : A}\ \textsf{lin-ax} \qquad \frac{}{\Gamma; \Delta, a : A; \vdash a : A}\ \textsf{aff-ax} \qquad \frac{i \in \mathbf{N}}{\Gamma, x : A; \Delta; \vdash x_i : A^\bullet}\ \textsf{exp-ax}$$

$$\frac{\Gamma; \Delta; \Sigma, l : A \vdash t : B}{\Gamma; \Delta; \Sigma \vdash \lambda l.t : A \multimap B}\ {\multimap}I \qquad \frac{\Gamma; \Delta; \Sigma \vdash t : A \multimap B \quad \Gamma; \Delta'; \Sigma' \vdash u : A}{\Gamma; \Delta, \Delta'; \Sigma, \Sigma' \vdash tu : B}\ {\multimap}E$$

$$\frac{\Gamma; \Delta; \Sigma \vdash t : A \quad \Gamma; \Delta'; \Sigma' \vdash u : B}{\Gamma; \Delta, \Delta'; \Sigma, \Sigma' \vdash t \otimes u : B}\ {\otimes}I \qquad \frac{\Gamma; \Delta; \Sigma \vdash u : A \otimes B \quad \Gamma; \Delta'; \Sigma', k : A, l : B \vdash t : C}{\Gamma; \Delta, \Delta'; \Sigma, \Sigma' \vdash \mathsf{let}\, k \otimes l = u \,\mathsf{in}\, t : C}\ {\otimes}E$$

$$\frac{\Gamma; \Sigma; \vdash t : A}{\Gamma; \Sigma; \vdash \bullet t : A^\bullet}\ {\bullet}I \qquad \frac{\Gamma; \Delta; \Sigma \vdash u : A^\bullet \quad \Gamma; \Delta', a : A; \Sigma' \vdash t : C}{\Gamma; \Delta, \Delta'; \Sigma, \Sigma' \vdash \mathsf{let}\, a^\bullet = u \,\mathsf{in}\, t : C}\ {\bullet}E$$

$$\frac{\ldots \quad \Gamma; ; \vdash \mathbf{u}(i) : A^\bullet \quad \ldots}{\Gamma; ; \vdash \mathbf{u} : A^\omega}\ \omega I \qquad \frac{\Gamma; \Delta; \Sigma \vdash u : A^\omega \quad \Gamma, x : A; \Delta'; \Sigma' \vdash t : C}{\Gamma; \Delta, \Delta'; \Sigma, \Sigma' \vdash \mathsf{let}\, x^\omega = u \,\mathsf{in}\, t : C}\ \omega E$$

Fig. 1. The simply-typed infinitary affine λ-calculus. In every non-unary rule we require that t, u (or, for the ωI rule, $\mathbf{u}(i), \mathbf{u}(j)$ for all $i \neq j \in \mathbf{N}$) contain pairwise disjoint sets of occurrences of the exponential variables in Γ.

(the abstraction binds all occurrences of x). Such a calculus may be embedded in the one introduced above, as follows:

$$x_i^\circ := \mathsf{let}\, a^\bullet = x_i \,\mathsf{in}\, a$$
$$(\lambda x.t)^\circ := \lambda l.\mathsf{let}\, x^\omega = l \,\mathsf{in}\, t^\circ$$
$$(t\langle u_0, u_1, u_2, \ldots\rangle)^\circ := t^\circ\langle \bullet u_0^\circ, \bullet u_1^\circ, \bullet u_2^\circ, \ldots\rangle$$

and we have $t \to t'$ implies $t^\circ \to^* t'^\circ$, so we do not lose generality. However, the categorical viewpoint adopted in the present paper naturally leads us to consider a simply-typed version of the calculus, given in Fig. 1. It is for this calculus that our construction provides denotational models. The types are generated by

$$A, B ::= X \mid A \multimap B \mid A \otimes B \mid A^\bullet \mid A^\omega,$$

where X is an atomic type. Note that the context of typing judgments has three *finite* components: exponential (Γ), affine (Δ) and linear (Σ). Although it may appear additive, the treatment of contexts is multiplicative also in the exponential case, as enforced by the condition in the caption of Fig. 1. The typing system enjoys the subject reduction property, as can be proved by an induction on the depth of the reduced redex.

3 Denotational Semantics

Definition 3 (reduced fpp, monoidal theory $\mathbb{P}$). *A* finite partial partition *(*fpp*) is a finite (possibly empty) sequence $(S_1, \ldots, S_k)$ of non-empty, pairwise disjoint subsets of $\mathbf{N}$. Fpp's may be composed as follows: let $\beta := (S_1, \ldots, S_k)$, with S_i infinite, and let $\beta' := (S'_1, \ldots, S'_{k'})$; we define $\beta' \circ_i \beta := (S_1, \ldots, S_{i-1}, T_1, \ldots, T_{k'}, S_{i+1}, \ldots, S_k)$, where each T_j is obtained as follows: let $n_0 < n_1 < n_2 < \cdots$ be the elements of S_i in increasing order; then,*

$T_j := \{n_m \mid m \in S'_j\}$. It must be noted that endowed with this composition, fpp's form an operad.

We will only consider reduced *fpp's, in which each S_i is either a singleton or infinite. We will use the notation $(S_1, \dots, S_m; T_1, \dots, T_n)$ to indicate that the S_i are singletons and the T_j are infinite, and we will say that such an fpp has size $m + n$. Note that the composition of reduced fpp's is reduced. The set of all reduced fpp's will be denoted by $\mathcal{P}$.*

Reduced fpp's induce a 2-sorted monoidal theory $\mathbb{P}$, as follows: each $\beta \in \mathcal{P}$ of size $m + n$ induces an arrow $\beta : (0, 1) \to (m, n)$ of $\mathbb{P}$. There is also an arrow $w : (1, 0) \to (0, 0)$ to account for partiality. Composition is defined as above.

For example, let $\beta := (E, O)$, where E and O are the even and odd integers, and let $\beta' := (\{0\}, \mathbf{N} \setminus \{0\})$ (these are actually total partitions). Then $\beta' \circ_1 \beta = (\{0\}, E \setminus \{0\}, O)$, whereas $\beta \circ_2 \beta' = (\{0\}, O, E \setminus \{0\})$.

Definition 4 (partitionoid). *A* partitionoid *in a symmetric monoidal category $\mathcal{C}$ is a strict symmetric monoidal functor*[5] *$G : \mathbb{P} \to \mathcal{C}$. Spelled out, it is a tuple $(G_0, G_1, w, (r_\beta)_{\beta \in \mathcal{P}})$ with (G_0, w) a copointed object and $r_\beta : G_1 \to G_0^{\otimes m} \otimes G_1^{\otimes n}$ whenever β is of size $m+n$, such that the composition of compatible w and r_β satisfies the equations induced by $\mathbb{P}$.*

A morphism of partitionoids $G \to G'$ is a pair of arrows $f_0 : G_0 \to G'_0$, $f_1 : G_1 \to G'_1$ such that f_0 is a morphism of copointed objects and $r'_\beta \circ f_1 = (f_0^{\otimes m} \otimes f_1^{\otimes n}) \circ r_\beta$ for all $\beta \in \mathcal{P}$ of size $m + n$.

We say that F is the free partitionoid on A *if it is endowed with an arrow $e : F_0 \to A$ such that, for every partitionoid G with an arrow $f : G_0 \to A$, there exists a unique morphism of partitionoids $(u_0, u_1) : G \to F$ such that $f = e \circ u$.*

For example, for any set X, $(X, X^{\mathbf{N}}, !_X, (r_\beta)_{\beta \in \mathcal{P}})$ is a partitionoid in **Set**, where $!_X$ is the terminal arrow $X \to 1$ and, if $\beta = (\{i_1\}, \dots, \{i_m\}; \{j_1^1 < j_2^1 < \cdots\}, \dots, \{j_1^n < j_2^n < \cdots\})$ and $f : \mathbf{N} \to X$, $r_\beta(f) := (f(i_1), \dots, f(i_m), k \mapsto f(j_k^1), \dots, k \mapsto f(j_k^n)) \in X^m \times (X^{\mathbf{N}})^n$.

Lemma 1. *If (F_0, F_1) is the free partitionoid on A, then $F_0 = A^\bullet$, the free co-pointed object on A.*

Proof. This follows from observing that $(A^\bullet, F_1)$ is also a partitionoid on A. $\square$

Definition 5 (infinitary affine category). *Let A be an object in a symmetric monoidal category. We denote by $\dagger_A$ the following diagram:*

$$\mathbf{1} \xleftarrow{\varepsilon_1} A^\bullet \xleftarrow{\varepsilon_2} (A^\bullet)^{\otimes 2} \longleftarrow \cdots \xleftarrow{\varepsilon_n} (A^\bullet)^{\otimes n} \xleftarrow{\varepsilon_{n+1}} (A^\bullet)^{\otimes n+1} \longleftarrow \cdots$$

where $\varepsilon_1 = \varepsilon$ is the copoint of $A^\bullet$ and $\varepsilon_{n+1} := (\mathrm{id})^{\otimes n} \otimes \varepsilon$, i.e., the arrow erasing the rightmost component. We set $A^\omega := \lim \dagger_A$ (if it exists).

An infinitary affine category *is a symmetric monoidal closed category such that, for all A, the free partitionoid on A exists and is $(A^\bullet, A^\omega)$.*

[5] An algebra for the fpp operad.

Several well-known categories are examples of affine infinitary categories: sets and relations, coherence spaces and linear maps, Conway games. Finiteness spaces are a non-example. We give the relational example here, which is a bit degenerate but easy to describe and grasp. For the others, we refer to Appendix B and Appendix C.

The category **Rel** has sets as objects and relations as morphisms. It is symmetric monoidal closed: the Cartesian product (which, unlike in **Set**, is not a categorical product in **Rel**!) acts both as $\otimes$ (with unit the singleton $\{*\}$) and $\multimap$. Let A be a set and let us assume that $* \notin A$. The free copointed object on A is (up to iso) $A \cup \{*\}$, with copoint the relation $\{(*, *)\}$. The F_1 part of the free partitionoid on A in **Rel** is (up to iso) the set of all functions $\mathbf{N} \to A^\bullet$ which are almost everywhere $*$. Given a reduced fpp $\beta := (\{i_1\}, \ldots, \{i_m\}; \{j_0^1 < j_1^1 < \ldots\}, \ldots, \{j_0^n < j_1^n < \ldots\})$, the corresponding morphism of type $A^\omega \to (A^\bullet)^{\otimes m} \otimes (A^\omega)^{\otimes n}$ is

$$r_\beta := \{(\mathbf{a}, (a_{i_1}, \ldots, a_{i_m}, \langle a_{j_0^1}, a_{j_1^1}, \ldots\rangle, \ldots, \langle a_{j_0^n}, a_{j_1^n}, \ldots\rangle)) \mid \mathbf{a} \in A^\omega\},$$

where we wrote $\langle a_0, a_1, a_2, \ldots\rangle$ for the function $\mathbf{a} : \mathbf{N} \to A^\bullet, i \mapsto a_i$.

Theorem 2. *An infinitary affine category is a denotational model of the infinitary affine λ-calculus.*

Proof. The interpretation of types is parametric in an assignment of an object to the base type X, and it is straightforward (notations are identical). In fact, we will confuse types and the objects interpreting them.

Let now $\Gamma; \Delta; \Sigma \vdash t : A$ be a typing judgment. The type of the corresponding morphism will be of the form $C_1 \otimes \cdots \otimes C_n \longrightarrow A$, where the C_i come from the context and are defined as follows. If it comes from $l : C \in \Sigma$ (resp. $a : C \in \Delta$), then $C_i := C$ (resp. $C_i := C^\bullet$). If it comes from $x : C \in \Gamma$, then $C_i := C^\omega$ if x appears infinitely often in t, otherwise, if it appears k times, $C_i := (C^\bullet)^{\otimes k}$.

The morphism interpreting a type derivation of $\Gamma; \Delta; \Sigma \vdash t : A$ is defined as customary by induction on the last typing rule. The $\mathsf{lin\text{-}ax}$ rule and all the rules concerning $\otimes$ and $\multimap$ are modeled in the standard way, using the symmetric monoidal closed structure. The only delicate point is modeling the seemingly additive behavior of the exponential context Γ in the binary rules (the same consideration will hold for the elimination rules of $\bullet$ and ω as well). Let us treat for instance the $\otimes I$ rule, and let us assume for simplicity that $\Gamma = x : C, y : D, z : E$, with x (resp. z) appearing infinitely often (resp. m and n times) in t and u, whereas y appears infinitely often in t but only k times in u. Let us also disregard the affine and linear contexts, which are unproblematic. The interpretation of the two derivations gives us two morphisms

$$[t] : C^\omega \otimes D^\omega \otimes (E^\bullet)^{\otimes m} \longrightarrow A, \qquad [u] : C^\omega \otimes (D^\bullet)^{\otimes k} \otimes (E^\bullet)^{\otimes n} \longrightarrow B.$$

Now, we seek a morphism of type $C^\omega \otimes D^\omega \otimes (E^\bullet)^{\otimes (m+n)} \longrightarrow A \otimes B$, because x and y appear infinitely often in $t \otimes u$, whereas z appears $m + n$ times. This is obtained by precomposing $[t] \otimes [u]$ with the morphisms $r_\beta : C^\omega \to C^\omega \otimes C^\omega$ and

$r_{\beta'} : D^\omega \to (D^\bullet)^{\otimes k} \otimes D^\omega$ associated with the fpp's $\beta = (; T_t, T_u)$ such that T_t (resp. T_u) contains all i such that x_i is free in t (resp. in u), and $\beta' = (S'_u; T'_t)$ is defined in a similar way with the variable y.

The weakening on exponential and affine variables in all axiom rules is modeled by the canonical morphisms $A^\bullet \to \mathbf{1}$ and $A^\omega \to \mathbf{1}$. For the rules $\mathsf{aff\text{-}ax}$ and $\mathsf{exp\text{-}ax}$, we use the canonical morphism $A^\bullet \to A$ and the identity on $A^\bullet$, respectively.

The $\bullet I$ rule is modeled by observing that objects of the form $\Gamma^\omega \otimes \Delta^\bullet$ are copointed (from tensoring their copoints), so from an arrow $\Gamma^\omega \otimes \Delta^\bullet \longrightarrow A$ we obtain a unique arrow $\Gamma^\omega \otimes \Delta^\bullet \longrightarrow A^\bullet$ by universality of $A^\bullet$. The $\bullet E$ rule is just composition.

For what concerns the ωI rule, let us assume for simplicity that $\Gamma = x : C$. This defines a sequence of objects $(C_i)_{i \in \mathbf{N}}$ such that C_i is either C^ω or $(C^\bullet)^{\otimes k_i}$ according to whether x appears in $\mathbf{u}(i)$ infinitely often or k_i many times. Let now $S_i := \{ j \in \mathbf{N} \mid x_j \text{ is free in } \mathbf{u}(i)\}$, define the fpp $\beta_i = (S_0, \ldots, S_i)$ and let

$$\varepsilon'_i := (\mathrm{id})^{\otimes i} \otimes w_i : C_0 \otimes \cdots \otimes C_{i-1} \otimes C_i \longrightarrow C_0 \otimes \cdots \otimes C_{i-1},$$

where $w_i : C_i \to \mathbf{1}$ is equal to $r_\emptyset$ if $C_i = C^\omega$ (with $\emptyset$ the empty fpp) or it is equal to $\varepsilon^{\otimes k_i}$ if $C_i = (C^\bullet)^{\otimes k_i}$. Let $\widehat{\beta}_i$ be the reduced fpp obtained from β_i by "splitting" its finite sets into singletons. If we set $\theta_i := r_{\widehat{\beta}_i}$, we have that for all $i \in \mathbf{N}$, $\varepsilon'_i \circ \theta_{i+1} = \theta_i$. Let now f_i be the interpretations of the derivations of $x : C; ; \vdash \mathbf{u}(i) : A^\bullet$ and consider the diagram

$$\begin{tikzcd}
 & & C^\omega \arrow[dll, "\theta_0"'] \arrow[dl, "\theta_1"] \arrow[d, "\theta_2"] \arrow[dr, "\theta_3"] \arrow[drr, "\theta_n"] & & \\
\mathbf{1} \arrow[d, "\mathrm{id}"] & C_0 \arrow[l, "\varepsilon'_0"] \arrow[d, "f_0"] & C_0 \otimes C_1 \arrow[l, "\varepsilon'_1"] \arrow[d, "f_0 \otimes f_1"] & C_0 \otimes C_1 \otimes C_2 \arrow[l, "\varepsilon'_2"] \arrow[d, "f_0 \otimes f_1 \otimes f_2"] & \cdots \arrow[l, "\varepsilon'_3"] \\
\mathbf{1} & A^\bullet \arrow[l, "\varepsilon_1"'] & (A^\bullet)^{\otimes 2} \arrow[l, "\varepsilon_2"'] & (A^\bullet)^{\otimes 3} \arrow[l, "\varepsilon_3"'] & \cdots \arrow[l, "\varepsilon_4"']
\end{tikzcd}$$

We showed above that all the upper triangles commute. It is easy to check that the bottom squares commute too, making $(C^\omega, ((f_0 \otimes \cdots \otimes f_{i-1}) \circ \theta_i)_{i \in \mathbf{N}})$ a cone for $\dagger_A$. Since $A^\omega = \lim \dagger_A$, this gives us a unique arrow $f : C^\omega \to A^\omega$, which we take as the interpretation of the derivation. The ωE rule is just composition, modulo the interposition of the canonical arrow $A^\omega \to (A^\bullet)^{\otimes k}$ in case x appears k times in t.

It remains to check that the above interpretation is stable under reduction, which may be done via elementary calculations. $\square$

4 Computing Symmetric Monoidal Kan Extensions

We mentioned that there is a well-known formula for computing regular Kan extensions (*i.e.* in **Cat**). This requires some notions coming from enriched category theory, which we recall next (although here the enrichment will be trivial, *i.e.* on **Set**).

Definition 6 (cotensor product of an object by a set). *Let $\mathcal{C}$ be a (locally small) category. Let A be an object in $\mathcal{C}$ and E a set. The* cotensor product $E \circ A$ *of A by E is defined by:*

$$\forall B \in \mathcal{C}, \mathcal{C}(B, E \circ A) \simeq \mathbf{Set}(E, \mathcal{C}(B, A))$$

Any locally small category with products is cotensored over **Set** *(all of its objects have cotensor products with any set) and the cotensor product is given by:*

$$E \circ A = \prod_{E} A$$

We will write $\langle f_e \rangle_{e \in E} : B \to E \circ A$ for the infinite pairing of arrows $f_e : B \to A$ and $\pi_e : E \circ A \to A$ the projections.

Definition 7 (end). *Let $\mathcal{C}$, $\mathcal{E}$ be two categories and $H : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{E}$ a functor. The* end *of H, denoted by $\int_{\mathcal{C}} H$, is defined as the universal object endowed with projections $\int_{\mathcal{C}} H \to H(c, c)$ for all $c \in \mathcal{C}$ making the following diagram commute:*

$$\begin{tikzcd}
\int_{c \in \mathcal{C}} H(c,c) \arrow[r] \arrow[d] & H(c',c') \arrow[d, "f^*"] \\
H(c,c) \arrow[r, "f_*"] & H(c,c')
\end{tikzcd}$$

for all arrows $f : c \to c'$ in $\mathcal{C}$.

Finally, here is the formula computing Kan extensions:

Theorem 3 ([9, X.4, Theorem 1]). *With the notations of Definition 2, whenever the objects exist:*

$$\mathrm{Ran}_K F(d) = \int_{c \in \mathcal{C}} \mathcal{D}(d, Kc) \circ Fc.$$

However, as mentioned in the introduction, the formula of Theorem 3 is only valid in **Cat** and we do not have any formula for computing a Kan extension in an arbitrary 2-category, or even in **SymMonCat**, our case of interest. Fortunately, Melliès and Tabareau proved a very general result [13, Theorem 1] giving sufficient conditions under which the Kan extension in **Cat** (something *a priori* worthless for our purposes) is actually the Kan extension in **SymMonCat** (what we want to compute). What follows is a specialized version of their result.

Theorem 4 ([13]). *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ be three symmetric monoidal categories and $F : \mathcal{C} \to \mathcal{E}$, $K : \mathcal{C} \to \mathcal{D}$ two monoidal symmetric functors. If (all the objects considered exist and) the canonical morphism*

$$X \otimes \int_{c \in \mathcal{C}} \mathcal{D}(d, Kc) \circ Fc \longrightarrow \int_{c \in \mathcal{C}} X \otimes \mathcal{D}(d, Kc) \circ Fc$$

is an isomorphism for every object X, then the right monoidal Kan extension (in the 2-category **SymMonCat***) of F along K may be computed as in Theorem 3.*

Proof. See Appendix A, Theorem 6. ∎

We may now give the abstract motivation behind Definition 5. The key property therein is that the free partitionoid on A is equal to $(A^\bullet, A^\omega)$. We now instantiate Theorem 4 to give a sufficient condition for that to be the case.

Proposition 1. *Let $\mathcal{C}$ be a symmetric monoidal closed category with all free partitionoids. If, for every objects X and A of $\mathcal{C}$, the canonical morphism*

$$X \otimes \int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}((0,1),(n,0)) \circ (A^\bullet)^{\otimes n} \longrightarrow \int_{n\in\mathbb{I}^{\mathrm{op}}} X \otimes \big(\mathbb{P}((0,1),(n,0)) \circ (A^\bullet)^{\otimes n}\big)$$

is an isomorphism, then $\mathcal{C}$ is an infinitary affine category.

Proof. In what follows, when denoting the objects of the theory $\mathbb{P}$, we use the abbreviation $n^\bullet := (n,0)$ and $n^\omega := (0,n)$.

Let A be an object of $\mathcal{C}$, seen as a strict monoidal functor $A : \mathbb{B} \to \mathcal{C}$. We let the reader check that, if $(A^\bullet, F_1)$ is the free partitionoid on A, then $F_1 = \mathrm{Ran}_{k'} A(1^\omega)$, where $k' : \mathbb{B} \to \mathbb{P}$ is the strict monoidal functor mapping $n \mapsto n^\bullet$ (indeed, Definition 4 is just this Kan extension spelled out). This functor may be written as $k \circ j$, with $j : \mathbb{B} \to \mathbb{I}^{\mathrm{op}}$ the inclusion functor and $k : \mathbb{I}^{\mathrm{op}} \to \mathbb{P}$ mapping $n \mapsto n^\bullet$, which induces a decomposition of the Kan extension, yielding $F_1 = \mathrm{Ran}_k A^\bullet(1^\omega)$. Now, the hypothesis is exactly the condition allowing us to apply Theorem 4, which gives us

$$F_1 = \int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet) \circ (A^\bullet)^{\otimes n},$$

so it is enough to prove that $\lim \dagger_A = \int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet) \circ (A^\bullet)^{\otimes n}$.

We start with showing that $\int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet) \circ (A^\bullet)^{\otimes n}$ is a cone for $\dagger_A$. Let $\psi_n : (0,1) \to (n,0)$ be the morphism corresponding to the fpp $(\{0\}, \ldots, \{n-1\};)$. By composing the canonical projection with π_{ψ_n} (see Definition 6) we get an arrow

$$p_n : \int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet) \circ (A^\bullet)^{\otimes n} \to \mathbb{P}(1^\omega, n^\bullet) \circ (A^\bullet)^{\otimes n} \to (A^\bullet)^{\otimes n}.$$

Observe now that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{P}(1^\omega, n^\bullet) \circ (A^\bullet)^{\otimes n} \arrow[r, "(\varepsilon_{n+1})^*"] \arrow[dr, bend right, "\pi_{\psi_n}"'] & \mathbb{P}(1^\omega, (n+1)^\bullet) \circ (A^\bullet)^{\otimes n} \arrow[d, "\pi_{\psi_{n+1}}"] \\
& (A^\bullet)^{\otimes n}
\end{tikzcd}$$

because $\varepsilon_{n+1} \circ \psi_{n+1} = \psi_n$. Moreover, the diagram

$$\begin{tikzcd}
\mathbb{P}(1^\omega, (n+1)^\bullet) \circ (A^\bullet)^{\otimes n+1} \arrow[r, "\pi_{\psi_{n+1}}"] \arrow[d, "(\varepsilon_{n+1})_*"] & (A^\bullet)^{\otimes n+1} \arrow[d, "\varepsilon_{n+1}"] \\
\mathbb{P}(1^\omega, (n+1)^\bullet) \circ (A^\bullet)^{\otimes n} \arrow[r, "\pi_{\psi_{n+1}}"] & (A^\bullet)^{\otimes n}
\end{tikzcd}$$

commutes too. So, by pasting them with the defining diagram of $\int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet)\circ (A^\bullet)^{\otimes n}$, one gets:

$$\begin{tikzcd}
\int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet)\circ (A^\bullet)^{\otimes n} \arrow[r] \arrow[d] & \mathbb{P}(1^\omega, (n+1)^\bullet)\circ (A^\bullet)^{\otimes n+1} \arrow[r, "\pi_{\psi_{n+1}}"] \arrow[d, "(\varepsilon_{n+1})_*"] & (A^\bullet)^{\otimes n+1} \arrow[dd, "\varepsilon_{n+1}"] \\
\mathbb{P}(1^\omega, n^\bullet)\circ (A^\bullet)^{\otimes n} \arrow[r, "(\varepsilon_{n+1})^*"] \arrow[drr, bend right, "\pi_{\psi_n}"'] & \mathbb{P}(1^\omega, (n+1)^\bullet)\circ (A^\bullet)^{\otimes n} \arrow[dr, "\pi_{\psi_{n+1}}"] & \\
 & & (A^\bullet)^{\otimes n}
\end{tikzcd}$$

In particular, $(\int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet)\circ (A^\bullet)^{\otimes n}, (p_n))$ is a cone for the diagram.

Reciprocally, let $(B, (b_n))$ be any cone for this diagram. (b_n) extends uniquely into a family (β_n) such that:

- $\forall n \in \mathbf{N}, b_n = \pi_{\psi_n} \circ \beta_n$
- (β_n) makes the following diagrams commute:

$$\begin{tikzcd}
B \arrow[r, "\beta_m"] \arrow[d, "\beta_n"] & \mathbb{P}(1^\omega, m^\bullet)\circ (A^\bullet)^{\otimes m} \arrow[d, "f_*"] \\
\mathbb{P}(1^\omega, n^\bullet)\circ (A^\bullet)^{\otimes n} \arrow[r, "f^*"] & \mathbb{P}(1^\omega, m^\bullet)\circ (A^\bullet)^{\otimes n}
\end{tikzcd}$$

for all $f : m \to n$ in $\mathbb{P}$.

Indeed, any element s of $\mathbb{P}(1^\omega, n^\bullet)$ is of the form $s = q \circ \psi_m$, where $m \geqslant n$ and $q \in \mathbb{I}^{\mathrm{op}}(m^\bullet, n^\bullet)$. So the family (β_n) is defined by:

$$\forall n \in \mathbf{N}, \beta_n = \langle A^\bullet(q)\circ b_m\rangle_{q\circ\psi_m\in\mathbb{P}(1^\omega, n^\bullet)}$$

is the unique family satisfying

$$\pi_{q\circ\psi_m}\circ\beta_n = q\circ\pi_{\psi_m}\circ\beta_m$$

This definition is sound, as $m > m'$ such that there exists $q, q', \psi_m, \psi_{m'}$ such that $s = q\circ\psi_m = q'\circ\psi_{m'}$, we have

$$q = q' \circ ((\mathrm{id})^{\otimes m'} \otimes (w^\bullet)^{\otimes m-m'})$$

and as such

$$A^\bullet(q) = A^\bullet(q')\circ\varepsilon_{m-m'+1}\circ\cdots\circ\varepsilon_m$$

and, as (b_n) is a cone for the sequential diagram,

$$A^\bullet(q)\circ b_m = A^\bullet(q')\circ b_{m'}.$$

So B makes the defining diagram of $\int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet)\circ (A^\bullet)^{\otimes n}$ commute, as such, (β_n) (and thus (b_n)) factors through it. Since all the cones of $\dagger_A$ factor through $\int_{n\in\mathbb{I}^{\mathrm{op}}} \mathbb{P}(1^\omega, n^\bullet)\circ (A^\bullet)^{\otimes n}$, it is its limit. $\square$

Observe that the condition of Proposition 1 is actually quite easy to grasp: it says that the limit of $\dagger_A$ commutes with the tensor, *i.e.*, if we denote by $X \otimes \dagger_A$ the $\dagger_A$ diagram in which each $(A^\bullet)^{\otimes n}$ and ε_n are replaced by $X \otimes (A^\bullet)^{\otimes n}$ and $\mathrm{id}_X \otimes \varepsilon_n$, respectively, then the condition says $\lim(X \otimes \dagger_A) = X \otimes \lim \dagger_A$.

5 From Infinitary Affine Terms to Linear Logic

In [10], it was shown that usual λ-terms may be recovered as *uniform* infinitary affine terms. The categorical version of this result is that, in certain conditions, a model of the infinitary affine λ-calculus is also a model of linear logic.

Theorem 5. *Let $\mathcal{C}$ be an infinitary affine category. If, for every objects X and A in $\mathcal{C}$, the canonical morphism*

$$X \otimes \int_{(n,m)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m} \longrightarrow \int_{(n,m)\in\mathbb{P}} X \otimes (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}$$

is an isomorphism, then $\mathcal{C}$ is a Lafont category. Moreover, the free commutative comonoid A^∞ on A may be computed as the equalizer of the diagram: where

$$\begin{tikzcd}
 & A^\omega & \\
A^\infty \arrow[r] & A^\omega \arrow[u, bend left, "(\varepsilon\otimes\mathrm{id})\circ\delta"] \arrow[u, bend right, "\mathrm{id}"'] \arrow[r, bend left, "(\mathrm{id}\otimes\delta)\circ\delta"] \arrow[r, bend right, "(\delta\otimes\mathrm{id})\circ\delta"'] \arrow[d, bend right, "\mathrm{swap}\circ\delta"'] \arrow[d, bend left, "\delta"] & (A^\omega)^{\otimes 3} \\
 & (A^\omega)^{\otimes 2} &
\end{tikzcd}$$

Fig. 2. Recovering the free co-commutative comonoid

$\delta : A^\omega \to A^\omega \otimes A^\omega$ and $\varepsilon : A^\omega \to \mathbf{1}$ are the morphisms induced by the fpp $(;E,O)$ (even and odd numbers) and the empty fpp, respectively, and $\mathrm{swap} : A^\omega \otimes A^\omega \to A^\omega \otimes A^\omega$ *is the symmetry of $\mathcal{C}$.*

Proof. Let $l : \mathbb{P} \to \mathbb{F}^{\mathrm{op}}$ be the strict monoidal functor mapping $(m,n) \mapsto m+n$ and collapsing every arrow $(0,1) \to (m,n)$ to the unique morphism $1 \to m+n$ in $\mathbb{F}^{\mathrm{op}}$. By composing Kan extensions, we know that $A^\infty = \mathrm{Ran}_l(A^\bullet, A^\omega)(1)$. Remark that $\mathbb{F}^{\mathrm{op}}(1,p)$ is a singleton for all $p \in \mathbf{N}$, so the hypothesis is exactly what allows to apply Theorem 4, giving us

$$A^\infty = \int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m} .$$

Now, $\int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}$ is the universal object making

$$\begin{tikzcd}
& \int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m} \arrow[dl, bend right, "\kappa_{n,m}"'] \arrow[dr, bend left, "\kappa_{n',m'}"] & \\
(A^\bullet)^{\otimes n} \otimes (A^\omega)^{\otimes m} \arrow[rr] & & (A^\bullet)^{\otimes n'} \otimes (A^\omega)^{\otimes m'}
\end{tikzcd}$$

commute. We are going to show that $\int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}$ is a cone for the diagram of Fig. 2. We will only show that

$$\begin{tikzcd}
& \int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m} \arrow[dl, bend right, "\delta \circ \kappa_{0,1}"'] \arrow[dr, bend left, "\mathrm{swap} \circ \delta \circ \kappa_{0,1}"] & \\
(A^\omega)^{\otimes 2} \arrow[rr] & & (A^\omega)^{\otimes 2}
\end{tikzcd}$$

commutes. The family $(\iota_n \otimes \iota_m \circ \delta \circ \kappa_{0,1})_{n,m}$ is a cone for $\dagger_A^{\otimes 2}$. Moreover, the $\theta_{n,m} \circ \delta$ are defined in terms of the operations of $\mathbb{P}$, they actually are the canonical maps, and

$$\forall n, m, \iota_n \otimes \iota_m \circ \delta \circ \kappa_{0,1} = \kappa_{0,n+m}$$

The exact same reasoning gives:

$$\forall n, m, \iota_n \otimes \iota_m \circ \mathrm{swap} \circ \delta \circ \kappa_{0,1} = \kappa_{0,n+m}$$

But $(\kappa_{0,n+m})_{n,m}$ factors uniquely through $(A^\omega)^{\otimes 2}$ (the limit of $\dagger_A^{\otimes 2}$) and as such,

$$\forall n, m, \delta \circ \kappa_{0,1} = \mathrm{swap} \circ \delta \circ \kappa_{0,1}$$

which is what we wanted. So $\int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}$ is a cone for the diagram of Fig. 2.

Let us now prove that every cone for the diagram of Fig. 2 is a cone of the diagrams defining $\int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}$.

It is easy to verify that any object B making the diagram defining A^∞ commute is endowed with exactly one map $B \to (A^\omega)^{\otimes n}$ for all $n \in \mathbf{N}$, built from δ and ε which, is moreover, stable under all swaps. In particular, by composing these maps $(B \to (A^\omega)^{\otimes n})_{n\in\mathbf{N}}$ with the arrow $A^\omega \to A^\bullet$, it is clear that there is a unique family of arrows

$$\forall n, m \in \mathbf{N}, B \to (A^\bullet)^{\otimes n} \otimes (A^\omega)^{\otimes m}$$

stable under extractions and weakenings. So any cone for the diagram defining A^ω is a cone for the diagram defining $\int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}$ and as such, factorizes through it. So $\int_{(m,n)\in\mathbb{P}} (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m}$ is the limit of the diagram of Fig. 2, and thus isomorphic to A^∞. $\square$

Intuitively, this construction amounts to collapsing the family of non-associative and non-commutative "contractions" built with δ, ε and swap.

It should be remarked that the particular δ used is not canonical, other morphisms would yield the same result. Indeed, from [10] we know that recovering usual λ-terms from infinitary affine terms is possible using uniformity which, as recalled in the introduction, amounts to identifying

$$\lambda x.\langle x_0, x_1, x_2, \dots\rangle \approx \lambda x.\langle x_{\beta(0)}, x_{\beta(1)}, x_{\beta(2)}, \dots\rangle,$$

for every injection $\beta : \mathbf{N} \to \mathbf{N}$. Theorem 5 amounts to defining a congruence on terms verifying

$$\lambda x.\, \langle x_0, x_1, x_2, \dots \rangle \simeq \lambda x.\, \langle x_0, x_2, x_4, \dots \rangle$$
$$\lambda x.\, \langle x_0, x_2, x_4, \dots \rangle \otimes \langle x_1, x_3, x_5, \dots \rangle \simeq \lambda x.\, \langle x_1, x_3, x_5, \dots \rangle \otimes \langle x_0, x_2, x_4, \dots \rangle$$

which is sufficient to recover $\approx$.

6 Discussion

We saw how the functorial semantic framework provides a bridge between the categorical and topological approaches to expressing the exponential modality of linear logic as a form of limit. This gives a way to construct, under certain hypotheses, denotational models of the infinitary affine λ-calculus. Moreover, it gives us a formula for computing the free exponential which is alternative to that of Melliès et al. Since both formulas apply only under certain conditions, it is natural to ask whether one of them is more general than the other. Although we do not have a general result, we are able to show that, under a mild condition verified in all models of linear logic we are aware of, our construction is applicable in every situation where Melliès et al.'s is.

Indeed, Melliès et al.'s construction amounts to checking that the Kan extension along m (below, left) is a monoidal Kan extension, whereas the one exposed in this article amounts to checking that the two Kan extensions along k, then l are monoidal (below, right):

$$\begin{tikzcd} & \mathcal{C} & \\ \mathbb{I}^{\mathrm{op}} \arrow[ur] \arrow[rr, "m"'] & & \mathbb{F}^{\mathrm{op}} \arrow[ul] \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{P} \arrow[d] \arrow[ddr, "l"] & \\ & \mathcal{C} & \\ \mathbb{I}^{\mathrm{op}} \arrow[uur, "k"] \arrow[ur] & & \mathbb{F}^{\mathrm{op}} \arrow[ul] \end{tikzcd}$$

As Kan extensions compose, it suffices to know that the Kan extension along m is monoidal, that $m = k \circ l$, and that there exists two monoidal natural transformations inside the two upper triangles that can be composed to the last one to be sure that the Kan extensions along k and along l are monoidal too. We thus get:

Proposition 2. *Let $\mathcal{C}$ be a symmetric monoidal category with all free partitionoids. Assume that Melliès et al.'s formula works and that A^ω exists. If there* *exists, for all integers n, m monoidal maps*

$$
\begin{aligned}
(A^\infty)^{\otimes n+m} \to & (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m} \\
& (A^\omega)^{\otimes n} \otimes (A^\bullet)^{\otimes m} \to (A^\bullet)^{\otimes n+m}
\end{aligned}
$$

that composed together are the $n+m$ tensor of the map $A^\infty \to A^{\leqslant 1} \to A^\bullet$ then $\mathcal{C}$ is an infinitary affine category and a Lafont category.

Actually, in all models we are aware of, either both formulas work, or neither does. For instance, our construction fails for finiteness spaces [4], as does the construction given in [14].

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An extended version of this work is available on the HAL open archive server.

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A Kan extensions through distributors

A.1 A distributive account of Kan extensions

The theory of distributors (or profunctors, or modules) provides a very elegant account of the Theorem 4 that extends to the Theorem 6. In the same way that functors generalize functions, distributors generalize relations between sets. In particular, any functor $\mathcal{A} \to \mathcal{B}$ can be seen as a distributor from $\mathcal{A}$ to $\mathcal{B}$ or in reverse, allowing to compose functors backwards.

Definition 8 (distributor). *Let $\mathcal{C}$ and $\mathcal{D}$ be two small categories. A* distributor *from $\mathcal{C}$ to $\mathcal{D}$ is a functor*

$$F : \mathcal{D}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$$

We will write it $F : \mathcal{C} \nrightarrow \mathcal{D}$. We will view it alternatively as a presheaf $\widehat{F} : \mathcal{D} \to \widehat{\mathcal{C}}$.

We say that a distributor $F : \mathcal{C} \nrightarrow \mathcal{D}$ is represented *by a functor $\phi : \mathcal{C} \to \mathcal{D}$ if F is the curryfication of the composite*

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{Y} \widehat{\mathcal{D}}$$

where $\widehat{\mathcal{D}}$ is the category of pre-sheafs over $\mathcal{D}$ and Y the Yoneda embedding. A distributor is said to be representable *if it is represented by a functor.*

Given three small categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$, and two distributors $F : \mathcal{C} \nrightarrow \mathcal{D}$, $G : \mathcal{D} \nrightarrow \mathcal{E}$, we define the composite $G \circ F : \mathcal{C} \nrightarrow \mathcal{E}$ by:

$$\forall c \in \mathcal{C}, \forall e \in \mathcal{E}, G \circ F(e, c) = \int^{d \in \mathcal{D}} G(e, d) \times F(d, c)$$

The composition is associative up to isomorphism (due to Fubini theorem for co-ends and the lax associativity of the cartesian product in **Set***). As such, we can define the bicategory* **Dist** *of distributors in the following way:*

- *objects are small categories;*
- *morphisms are distributors with the aforementioned composition;*
- *2-morphisms are natural transformation.*

The bicategory **Dist** enjoys a close relationship with the bicategory **Cat** of small categories. Indeed, any functor $F : \mathcal{C} \to \mathcal{D}$ induce a pair of adjoints distributors $F_* \dashv F^*$ in the following way:

$$F_* : \mathcal{C} \nrightarrow \mathcal{D} \text{ defined by } F_*(d, c) = \mathcal{D}(d, Fc)$$
$$F^* : \mathcal{D} \nrightarrow \mathcal{C} \text{ defined by } F^*(c, d) = \mathcal{D}(c, Fd)$$

Moreover, the functor $\cdot \mapsto (\cdot)_* : \mathbf{Cat} \to \mathbf{Dist}$ is locally fully faithful, that is,

$$\mathbf{Cat}(f, g) \simeq \mathbf{Dist}(f_*, g_*)$$

This situation between **Cat** and **Dist** is called a *pro-arrow equipment* [15]. It is a actually the prototypical exemple. All that follows is adaptable in the general case of a pro-arrow equipment between two arbitrary bicategories.

The following lemma is folklore:

Lemma 2. *Let $f : \mathcal{C} \to \mathcal{D}$ and $j : \mathcal{C} \to \mathcal{E}$ be two functors. The representative of the distributor $f_* \circ j^*$ (if it exists) is the left Kan extension of f along j.*

Proof. Let $g : \mathcal{E} \to \mathcal{D}$. We will note $\mathrm{Lan}_j f$ the representative of $f_* \circ j^*$.

$$\begin{aligned}\mathbf{Cat}(\mathrm{Lan}_j f, g) &\simeq \mathbf{Dist}(f_* \circ j^*, g_*)\\ &\simeq \mathbf{Dist}(f_*, g_* \circ j_*)\\ &\simeq \mathbf{Dist}(f_*, (g \circ j)_*)\\ &\simeq \mathbf{Cat}(f, g \circ j)\end{aligned}$$

Corollary 1. *Let $f : \mathcal{C} \to \mathcal{D}$ and $j : \mathcal{C} \to \mathcal{E}$ be two functors. If it exists, the coend*

$$\int^{c \in \mathcal{C}} \mathcal{E}(j(c), ^-) \otimes f(c)$$

is the left Kan extension of f along j. Dually,

$$\int_{c \in \mathcal{C}} \mathcal{E}(^-, j(c)) \circ f(c)$$

is the right Kan extension of f along j.

Proof. Let $g : \mathcal{E} \to \mathcal{D}$ be any functor.

$$\begin{aligned}&\mathbf{Dist}\,(f_* \circ j^*, g_*)\\
\simeq&\mathbf{Dist}\left((d,e) \mapsto \int^c \mathcal{E}(jc,e) \times \mathcal{D}(d, fc), (d,e) \mapsto \mathcal{D}(d, ge)\right)\\
\simeq&\int_e \widehat{\mathcal{D}}\left(d \mapsto \int^c \mathcal{E}(jc,e) \times \mathcal{D}(d, fc), d \mapsto \mathcal{D}(d, ge)\right), \text{by MacLane's parameter's theorem}\\
\simeq&\int_e \int_c \widehat{\mathcal{D}}\left(d \mapsto \mathcal{E}(jc,e) \times \mathcal{D}(d, fc), d \mapsto \mathcal{D}(d, ge)\right), \text{by continuity of the Hom functor}\\
\simeq&\int_e \int_c \widehat{\mathcal{D}}\left(\mathcal{E}(jc,e) \otimes d \mapsto \mathcal{D}(d, fc), d \mapsto \mathcal{D}(d, ge)\right)\\
\simeq&\int_e \int_c \mathbf{Set}\left(\mathcal{E}(jc,e), \widehat{\mathcal{D}}(d \mapsto \mathcal{D}(d, fc), d \mapsto \mathcal{D}(d, ge))\right)\\
\simeq&\int_e \int_c \mathbf{Set}\left(\mathcal{E}(jc,e), \mathcal{D}(fc, ge))\right), \text{by the Yoneda lemma}\\
\simeq&\int_e \int_c \mathcal{D}(\mathcal{E}(jc,e) \otimes fc, ge)\\
\simeq&\int_c \mathbf{Cat}(e \mapsto \mathcal{E}(jc,e) \otimes fc, g), \text{by Fubini's theorem and MacLane's parameter theorem}\\
\simeq&\mathbf{Cat}\left(\int_c e \mapsto \mathcal{E}(jc,e) \otimes fc, g\right), \text{by continuity of the Hom functor}\end{aligned}$$

So, the functor

$$\int_c e \mapsto \mathcal{E}(jc,e) \otimes fc$$

is the representative of $f_* \circ j^*$. It is the left Kan extension of f along j.

A.2 The symmetric monoidal case

Definition 9 (discrete fibration). *A functor $F : \mathcal{C} \to \mathcal{B}$ is a* discrete fibration *if for every object c in $\mathcal{C}$, and every morphism $g : b \to Fc$ in $\mathcal{B}$, there is a unique morphism $h : d \to c$ in $\mathcal{C}$ such that $Fh = g$.*

Definition 10 (category of elements). *Let $F : \mathcal{C} \to \mathbf{Set}$. The* category of elements $\mathrm{Elt}(F)$ *is the category whose objects are pairs (c, x) where c is an object of $\mathcal{C}$ and $x \in Fc$, and morphisms $(c, x) \to (c', x')$ are morphisms $u : c \to c'$ such that $P(u)(x) = x'$.*

Lemma 3. *A functor $F : \mathcal{C} \to \mathcal{B}$ is a discrete fibration if and only if $\mathcal{C}$ is isomorphic to the category of elements $\mathrm{Elt}(\varphi)$ of a presheaf over $\mathcal{B}$, and, up to the isomorphism, F is the first projection.*

Proof. Let F be a discrete fibration. Let $\varphi : \mathcal{B}^{\mathrm{op}} \to \mathbf{Set}$ defined on the objects by $\varphi b = \{c \in \mathcal{C}, Fc = b\}$; and the image of a map $g : b \to b'$ is defined as the function $f : \varphi b' \to \varphi b$ that maps $c \in \varphi b'$ to the unique $d \in \mathcal{C}$ such that g is the image of a map $d \to c$.

The other direction is trivial.

Every functor factors essentially uniquely as a composite of a final functor and a discrete fibration. As such, any functor $\mathcal{B} \to \mathcal{C}$ can be seen as a presheaf over $\mathcal{C}$. This allows to reify diagrams in $\mathcal{C}$ as presheafs over $\mathcal{C}$; which offers a language to formalize the fact that diagrams of a certain shape have colimits.

Consider a class $\mathcal{F}$ of diagram shapes (that is, of small categories), containing $\mathbf{1}$. The category $\mathcal{C}^{\mathcal{F}}$ of the diagrams of shape $\mathcal{F}$ in $\mathcal{C}$ corresponds to a subcategory $\overline{\mathcal{C}}$ of $\widehat{\mathcal{C}}$ by the aforementioned correspondance. As $\mathcal{F}$ contains $\mathbf{1}$, the Yoneda embedding $\mathcal{C} \to \widehat{\mathcal{C}}$ restricts to

$$\begin{aligned} y : \mathcal{C} &\to \overline{\mathcal{C}} \\ c &\mapsto (\mathbf{1} \mapsto c) \end{aligned}$$

We can reason about the properties of this embedding. As a restriction of the Yoneda embedding, it is fully faithful. Moreover, for every category $\mathcal{A}$ and functor $f = \mathcal{A} \to \overline{\mathcal{C}}$, a computation based on the Yoneda lemma shows that

$$\mathbf{Dist}(y^* \circ f_*, y^* \circ g_*) = \mathbf{Cat}(f, g)$$

If it exists, the functor colim : $\overline{\mathcal{C}} \to \mathcal{C}$ that associates to every diagram in $\overline{\mathcal{C}}$ its colimit is the left adjoint of y. So, the fact that the diagrams of a certain shape have a colimit in $\mathcal{C}$ can be formally expressed as the fact that the Yoneda embedding associated to that shape has a left adjoint; the fact that the colimits commute with the tensor product can be expressed as the fact that the left adjoint is symmetric monoidal.

Lemma 4. *Let $f : \mathcal{B} \nrightarrow \mathcal{C}$ be a distributor and $\overline{\mathcal{C}}$ be a full subcategory of $\widehat{\mathcal{C}}$ (the presheaf category over $\mathcal{C}$) containing $\mathcal{C}$. Suppose that the Yoneda embedding of $\mathcal{C}$ into $\overline{\mathcal{C}}$ has a left adjoint*

$$\mathrm{colim} \dashv y : \overline{\mathcal{C}} \to \mathcal{C}$$

and that f factors through y^* as

$$\mathcal{B} \xrightarrow{\overline{f}_*} \overline{\mathcal{C}} \xrightarrow{y^*} \mathcal{C}$$

where $\overline{f} : \mathcal{B} \to \overline{\mathcal{C}}$ *is a functor. The functor* $\mathrm{colim} \circ \overline{f}$ *is a representative of* f*.*

Proof.

$$\begin{aligned}
\mathbf{Dist}(f, g^*) &= \mathbf{Dist}(y^* \circ \overline{f}_*, g_*) \\
&\simeq \mathbf{Dist}(y^* \circ \overline{f}_*, y^* \circ y_* \circ g_*), \text{ as the Yoneda embedding is fully faitfull} \\
&\simeq \mathbf{Cat}(\overline{f}, y \circ g) \\
&\simeq \mathbf{Cat}(\mathrm{colim} \circ \overline{f}, g)
\end{aligned}$$

Definition 11. *Let* $T : \mathbf{Cat} \to \mathbf{Cat}$ *be the monad associating to any category its free symmetric monoidal category. If* A *is an algebra over* T*, we will write* $[] : TA \to A$ *the structure map.*

A symmetric monoidal distributor *is a distributor* F *making*

$$\begin{array}{ccc}
TA & \xrightarrow{TF} & TB \\
\downarrow{\scriptstyle [-]^*} & & \downarrow{\scriptstyle [-]^*} \\
A & \xrightarrow{F} & B
\end{array}$$

commute for all $F : A \nrightarrow B$*.*

Lemma 5. *Let* $j : \mathcal{A} \to \mathcal{B}$ *be a monoidal symmetric functor between two monoidal symmetric categories.* j^* *is symmetric monoidal if and only if*

$$\int^{A \in T\mathcal{A}} \mathcal{A}(a, [A]) \times T\mathcal{B}(TjA, B) \to \mathcal{B}(ja, B)$$

is an isomorphism.

Theorem 6 (Melliès and Tabareau [13]). *Let* $\mathcal{A}, \mathcal{B}, \mathcal{C}$ *be symmetric monoidal categories.*

Suppose that the Yoneda embedding $y : \mathcal{C} \to \overline{\mathcal{C}}$ *has a left adjoint* $\mathrm{colim} \dashv y$ *and that both* colim *and* y *are symmetric monoidal.*

Let $j : \mathcal{A} \to \mathcal{B}$ *be a symmetric monoidal functor such that* j^* *is a symmetric monoidal distributor, and* $f : \mathcal{A} \to \mathcal{C}$ *be a symmetric monoidal functor. If the distributor* $f_* \circ j^*$ *factors through* y^* *as:*

$$\mathcal{B} \xrightarrow{g_*} \overline{\mathcal{C}} \xrightarrow{y^*} \mathcal{C}$$

then the forgetful functor

$$U_j : \mathbf{SymMonCat}(\mathcal{B}, \mathcal{C}) \to \mathbf{SymMonCat}(\mathcal{A}, \mathcal{C})$$

has a left adjoint

$$\mathrm{Lan}_j : \mathbf{SymMonCat}(\mathcal{A}, \mathcal{C}) \to \mathbf{SymMonCat}(\mathcal{B}, \mathcal{C})$$

computed with a Kan extension.

Proof. The functor $\mathrm{colim} \circ g$ is the Kan extension of f along j: by Lemma 4, it is the representative of $f_* \circ j^*$, and by Lemma 2, the representative of $f_* \circ j^*$ is the Kan extension of f along j.

It remains to check that it is symmetric monoidal. As colim is (by hypothesis), it suffices to show that g is. By hypothesis, $f_* \circ j^*$ is symmetric monoidal. As composition with y^* induces a fully faithful functor, g is symmetric monoidal. The same argument gives the functoriality of the construction. □

B Examples of Infinitary Affine Categories

We will describe the construction in models of Multiplicative-Additive Linear Logic where Melliès et al.'s construction is known to work.

B.1 The relational model

The relational model is the simplest possible model of linear logic and is completely degenerate: all the dual operations are identified. As such, we will not verify the conditions we laid out, and just describe the results.

Definition 12 (Rel). *The category* **Rel** *is the category with objects the sets and with morphisms the relations between them.*

It is a model of MALL with any singleton as a dualizing object, cartesian product of sets as tensor product and disjoint union as cartesian product.

Rel has furthermore arbitrary products (not just finite). Let's apply our recipe. Let A be an object in **Rel**. For notation convenience, we will write $\mathbf{1} = \{*\}$.

Step 1 The free co-pointed object under A is given by $A^\bullet = A \,\&\, \mathbf{1}$.

Step 2 We remark that the weakening arrows $\varepsilon_n : (A^\bullet)^{\otimes n} \to (A^\bullet)^{\otimes n-1}$ arrows are defined by:

$$\varepsilon_n = \{((a_0, \dots, a_n, *), (a_0, \dots, a_n)) \mid \forall 0 \leqslant i \leqslant n, a_i \in A \,\&\, \mathbf{1}\}.$$

A^ω is the set of almost finite sequences of elements of $A^\bullet$, that is functions $\mathbf{N} \to A^\bullet$ which are almost everywhere $*$. Given a reduced fpp $\beta := (\{i_1\}, \dots, \{i_m\}; \{j_0^1 < j_1^1 < \dots\}, \dots, \{j_0^n < j_1^n < \dots\})$, the corresponding morphism of type $A^\omega \to (A^\bullet)^{\otimes m} \otimes (A^\omega)^{\otimes n}$ is

$$r_\beta := \{(\mathbf{a}, (a_{i_1}, \dots, a_{i_m}, \langle a_{j_0^1}, a_{j_1^1}, \dots \rangle, \dots, \langle a_{j_0^n}, a_{j_1^n}, \dots \rangle)) \mid \mathbf{a} \in A^\omega\},$$

where we wrote $\langle a_0, a_1, a_2, \dots \rangle$ for the function $\mathbf{a} : \mathbf{N} \to A^\bullet$.

Step 3 The commutation relations are satisfied and the free co-commutative comonoid under A is computed as the set of finite multisets over A:

$$A^\infty = \mathcal{M}_{\mathrm{fin}}(A)$$

Indeed, if we define the *multi-support* of a sequence $\boldsymbol{a} \in A^\omega$ co-inductively as

$$\mathrm{msupp}(a_0, a_1, \dots) = \begin{cases} [a_0] + \mathrm{msupp}(a_1, a_2, \dots) & \text{if } a_0 \neq * \\ \mathrm{msupp}(a_1, a_2, \dots) & \text{else} \end{cases}$$

As discussed, asking two sequences to be equalized amounts to asking they have the same multi-support. The limiting cone is defined by

$$\iota_n = \{\mu \multimap \boldsymbol{a}, \mu = \mathrm{msupp}(\boldsymbol{a})\}$$

So, the construction is valid in the relational model.

B.2 Coherence spaces

Coherence spaces are the original semantics of linear logic, from which it sprung.

Definition 13 (Coh, [5, section 3]). *A* coherence space *is a pair $A = (|A|, \frown_A)$ of a set $|A|$ (the web of A) and a reflexive binary relation $\frown_A$ on the elements of $|A|$, the coherence of A. A clique of A is a set of pairwise coherent elements of $|A|$.*

Every coherence space A has a dual coherence space $A^\perp$ with same web $|A|$ and coherence relation

$$a \frown_{A^\perp} b \Leftrightarrow a = b \text{ or } \neg(a \frown_A b)$$

The coherence of the dual is called the incoherence of the primal and written $a \asymp_A b$.

For every coherence spaces A and B, we define their tensor product as the coherence space $A \otimes B$ with web $|A| \times |B|$ and coherence defined by

$$a \otimes b \frown_{A \otimes B} a' \otimes b' \Leftrightarrow a \frown_A a' \text{ and } b \frown_B b'$$

where $a \otimes b$ is just a semantically-flavoured notation for the couple (a, b).

For every coherence spaces A and B, we define their cartesian product as the coherence space $A \,\&\, B$ with web $|A| + |B|$ and coherence defined by

$$\begin{aligned} a &\frown_{A\&B} a' \Leftrightarrow a \frown_A a' \\ b &\frown_{A\&B} b' \Leftrightarrow b \frown_B b' \\ a &\frown_{A\&B} b \end{aligned}$$

The category **Coh** *is defined as the category with coherence spaces as objects and cliques of $A \multimap B = (A \otimes B^\perp)^\perp$ as morphisms. For clarity's sake, we explicit the coherence of $A \multimap B$.*

$$a \multimap b \frown_{A \multimap B} a' \multimap b' \Leftrightarrow \begin{cases} a \frown_A a' \Rightarrow b \frown_B b' \\ a \asymp_A a' \Rightarrow b \asymp_B b' \end{cases}$$

where $a \multimap b$ is again a notation for (a, b). There is a forgetful functor from cliques of $A \multimap B$ to relations of A and B. As such, cliques compose as relations.

It is a $*$-autonomous category with finite products, and thus a model of MALL.

Let's apply our recipe. Let A be an object of **Coh**.

Step 1 The category **Coh** has finite products. As such, we can define the free co-pointed object $A^\bullet$ to be equal to $A \,\&\, \mathbf{1}$ equipped with its second projection.

Step 2 We claim that A^ω is defined as follows:

- its web $|A^\omega|$ is the set of almost finite sequences of elements of $A^\bullet$, $\mathbf{N} \to A^\bullet$ which are almost everywhere $*$.
- given two sequences $\boldsymbol{a}$ and $\boldsymbol{b}$,

$$\boldsymbol{a} \frown_{A^\omega} \boldsymbol{b} \Leftrightarrow \forall i \in \mathbf{N}, \text{if } (a_i \in |A| \text{ and } b_i \in |A|), \text{ then } a_i \frown_A b_i$$

- given a reduced fpp

$$\beta := (\{i_1\}, \dots, \{i_m\}; \{j_0^1 < j_1^1 < \dots\}, \dots, \{j_0^n < j_1^n < \dots\}),$$

  the projecting cone is

$$r_\beta := \{\mathbf{a} \multimap (a_{i_1} \otimes \dots \otimes a_{i_m} \otimes \langle a_{j_0^1}, a_{j_1^1}, \dots\rangle \otimes \dots \otimes \langle a_{j_0^n}, a_{j_1^n}, \dots\rangle) \mid \mathbf{a} \in A^\omega\},$$

  where we wrote $\langle a_0, a_1, a_2, \dots\rangle$ for the function $\mathbf{a} : \mathbf{N} \to A^\bullet$.

For A^ω to be well-defined, we only have to prove that for all $n \in \mathbf{N}$, ι_n is indeed a clique of $A^\omega \multimap (A^\bullet)^{\otimes n}$. Let

$$\mathbf{a} \multimap (a_{i_1} \otimes \dots \otimes a_{i_m} \otimes \langle a_{j_0^1}, a_{j_1^1}, \dots\rangle \otimes \dots \otimes \langle a_{j_0^n}, a_{j_1^n}, \dots\rangle)$$

and

$$\mathbf{b} \multimap (b_{i_1} \otimes \dots \otimes b_{i_m} \otimes \langle b_{j_0^1}, b_{j_1^1}, \dots\rangle \otimes \dots \otimes \langle b_{j_0^n}, b_{j_1^n}, \dots\rangle)$$

be two elements of r_β. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are coherent, then for all index i, $a_i \frown b_i$ and

$$\begin{aligned}&(a_{i_1} \otimes \dots \otimes a_{i_m} \otimes \langle a_{j_0^1}, a_{j_1^1}, \dots\rangle \otimes \dots \otimes \langle a_{j_0^n}, a_{j_1^n}, \dots\rangle)\\ \frown&(b_{i_1} \otimes \dots \otimes b_{i_m} \otimes \langle b_{j_0^1}, b_{j_1^1}, \dots\rangle \otimes \dots \otimes \langle b_{j_0^n}, b_{j_1^n}, \dots\rangle),\end{aligned}$$

else they are incoherent, that is, for a certain $i \leqslant n$ (as for all indices greater than a certain n, the two sequences coincide) $a_i \asymp b_i$, which is absurd. So r_β is a clique.

To show that A^ω is the limit of the diagram, we have to check that it is a cone (which is obvious) and that it is universal among them. Let $(B, (b_\beta))$ be another cone. The relation

$$\{x \multimap (y_0, y_1, \dots), \forall x \in |B|, \forall i \in \mathbf{N}, (x \multimap (y_{i_1} \otimes \dots \otimes y_{i_m} \otimes \langle y_{j_0^1}, \dots\rangle \otimes \dots) \in b_\beta\}$$

is well-defined, a clique and is a relation between B and A^ω that factorizes the (b_n). It is moreover unique as any other factorizing relation would have to coincide with the composition with the projections. So A^ω is the limit of the diagram.

Moreover, for all object X, $X \otimes A^\omega$ is the limit of the tensorized diagram, as the coherence in the two parts of a tensor product are independent. So A^ω is indeed the right Kan extension of A.

Step 3 Given the remark at the end of Section 5, the construction makes the order collapse and requires all the elements to be pairwise coherent. As such, it is easy to check that the limit of the diagram of Theorem 5 is A^∞, the space with web the set of finite multicliques (multisets of pairwise coherent elements) over A and coherence defined by:

$$\forall \mu, \nu \in |A^\infty|, \mu \supset \nu \Leftrightarrow \mu + \nu \in |A^\infty|$$

and $e : A^\infty \to A^\omega$ defined by $e = \{\mu \multimap \boldsymbol{a}, \mu$ is the multi-support of $\boldsymbol{a}, \mu \in |A^\omega|\}$.

The conditions of proposition 5 are verified, and so $!A = A^\infty$ is the free co-commutative comonoid under A, with co-multiplication (contraction):

$$\left\{\mu \multimap \nu \otimes \rho \in |!A| \times |!A|^2, \nu + \rho = \mu\right\}.$$

B.3 Conway games

Definition 14 (Conway$^-$). *A Conway game A is an oriented rooted well-founded graph (V_A, E_A, λ_A) consisting of a set V_A of vertices (the positions of the game), a set $E_A \subseteq V_A \times V_A$ of edges (the moves of the game), and a function $\lambda_A : E_A \to \{-1; +1\}$ indicating wether a move is played by Opponent (-1) or Proponent $(+1)$. We write $\star_A$ for the root of the underlying graph. A Conway game is called negative when all its moves starting from the root are played by Opponent.*

A play $s = m_1 \cdot m_2 \cdot \dots \cdot m_k$ of a Conway game A is a path $s : \star_A \twoheadrightarrow x_k$ starting from the root $\star_A$ whose vertices are labeled by $m_1, \dots, m_k$. A play is alternating when $\forall i \in \{1, \dots, k-1\}, \lambda_A(m_{i+1}) = -\lambda_A(m_i)$.

Every Conway game A induces a dual game $A^\perp$ by reversing the polarities of the moves.

For every Conway games A and B, we define their tensor product as the game $A \otimes B$ with set of vertices $V_{A\otimes B} = V_A \times V_B$, moves defined as

$$x \otimes y \to \begin{cases} z \otimes y \text{ if } x \to z \text{ in the game } A \\ x \otimes z \text{ if } y \to z \text{ in the game } B \end{cases}$$

and the polarity of a move in $A \otimes B$ defined as the polarity of the underlying move in the component A or the component B. The Conway game $\mathbf{1}$ with a unique position $\star$ and no moves is the neutral element of the tensor product.

For every negative Conway games A and B, we define their cartesian product $A \,\&\, B$ as the game whose

– *set of positions is the amalgamated sum of the positions of A and B under the initial position;*
– *Opponent moves from the initial position $\star_{A\&B}$ are of two kinds*

$$\star_{A\&B} \to \begin{cases} x \text{ if } \star_A \to x \text{ in the game } A \\ y \text{ if } \star_B \to y \text{ in the game } B \end{cases}$$

– *moves from a position x in the component A (respectively B) are exactly the moves from x in the Conway game A (respectively B), with the same polarity.*

A strategy of a Conway game A is defined as a non empty set of alternating plays of even length such that every non-empty play starts with an Opponent move, σ is closed by even length prefix, and σ is deterministic, that is, for all plays s and all moves m, n, n',

$$s \cdot m \cdot n \in \sigma \wedge s \cdot m \cdot n' \Rightarrow n = n'$$

The category **Conway**$^-$ *has negative Conway games as objects and strategies σ of $A^\perp \otimes B$ as morphisms $\sigma : A \to B$. It is symmetric monoidal closed and has finite and infinite products. As such, it is a model (degenerate, as it is compact closed) of the multiplicative and additive fragment of linear logic.*

Step 1 The monoidal unit $\mathbf{1}$ is terminal in the category **Conway**$^-$. As such, the free affine object $A^\bullet$ is just the object A itself.

Step 2 The limit A^ω is the game defined as follows:

– its positions are infinite words $w = x_1 \cdots x_n \cdots$ whose letters are positions x_i of the game A such that only a finite number of them are different than $\star_A$,
– its root is the word $\star_A \cdots \star_A \cdots$,
– a move $w \to w'$ is a move played in one copy:

$$w_1 x w_2 \to w_1 y w_2$$

– the polarities are inherited from those of A.

It is endowed with projections $A^\omega \to A^{\otimes n}$ the partial copycat strategies playing only in the first n letters.

The commutation condition is verified: they follow from [14].

Step 3 The equalizer A^∞ is defined as follows:

– its positions are infinite words $w = x_1 \cdots x_n \cdots$ whose letters are positions x_i of the game A such that for every $i \in \mathbf{N}^\star$, the position x_{i+1} is the root $\star_A$ whenever the position x_i is the root $\star_A$,
– the root is the word $\star_A^\omega$,
– a move is a move played in one copy,

– the polarities are inherited from those of A.

It is equipped with the equalizer $A^\infty \to A^\omega$ consisting of copycat reordering strategies.

The construction on the category of negative Conway games yields the construction for the whole category of Conway games. For details, see [14].

C A counter-example: finiteness spaces

Definition 15 (Fin, [4]). *Let E be a countable set. Two subsets u and $u' \subseteq E$ are orthogonal (written $u \perp u'$) when their intersection $u \cap u'$ is finite.*

The orthogonal of a family $\mathcal{F}$ of subsets of E is then defined as

$$\mathcal{F}^\perp = \{u \in E | \forall v \in \mathcal{F}, \#(u \cap v) < \aleph_0\}$$

A finiteness space *is a couple $(|E|, \mathsf{F}(E))$ of a set $|E|$, the web of E and $\mathsf{F}(E) \subseteq \mathfrak{P}(E)$ the finiteness of E, a family verifying $\mathsf{F}(E)^{\perp\perp} = \mathsf{F}(E)$. The elements of $\mathsf{F}(E)$ are called finitary, the ones of $\mathsf{F}(E)^\perp$ are called antifinitary.*

For every finiteness space E, we define its dual by $E^\perp = (|E|, \mathsf{F}(E)^\perp)$.

For every finiteness spaces E and F, we define their tensorial product as the space $E \otimes F$ with web $|E| \times |F|$ and finiteness defined by:

$$\mathsf{F}(E \otimes F) = \{w \subseteq |E| \times |F|, \pi_E(w) \in \mathsf{F}(E), \pi_F(w) \in \mathsf{F}(F)\}$$

where π_E and π_F are the usual projections. The unit of the tensor product is defined by $|\mathbf{1}| = \{*\}$ and $\mathsf{F}(\mathbf{1}) = \{\varnothing, \{*\}\}$.

For every finiteness spaces E and F, we define their cartesian product as the space $E \,\&\, F$ with web $|E| \sqcup |F|$ and finiteness $\mathsf{F}(E) \sqcup \mathsf{F}(F)$.

We say that a relation $R \subseteq |E| \times |F|$ between two finiteness spaces E and F is finitary if

$$\forall u \in \mathsf{F}(E), R(u) = \{f \in |F|, \exists e \in u, eRf\} \in \mathsf{F}(F)$$
$$\forall v \in \mathsf{F}(F)^\perp, {}^tR(v) = \{e \in |E|, \exists f \in u, eRf\} \in \mathsf{F}(E)^\perp$$

The linear implication $E \multimap F$ is defined as the space with web $|E| \times |F|$ and finiteness structure $\mathsf{F}(E \multimap F)$ the set of finitary relations between E and F.

*The category **Fin** of finiteness spaces has finiteness spaces as objects and finitary relations as morphisms. It is a $*$-autonomous category with finite products and thus a model of the multiplicative and additive fragment of linear logic.*

Step 1 The category **Fin** has finite products. As such, we define $A^\bullet = A \,\&\, \mathbf{1}$.

Step 2 A^ω is the following object: its web is the set of almost finite sequences of elements of $A^\bullet$, that is functions $\mathbf{N} \to A^\bullet$ which are almost everywhere $*$; its finiteness structure is

$$\mathsf{F}(A^\omega) = \{x \subseteq |A^\omega|, \forall n \in \mathbf{N}, \forall i \leqslant n, \pi_i(x) \in \mathsf{F}(A^\bullet)\}$$

where $x \,|\, n = \{a \in x, \forall i \geqslant n, a(i) = *\}$ and π_i is the i-th projection.

This is not the free partitionoid on A, as its finiteness structure is given by:

$$\{x \subseteq |A^\omega|, \forall n \in \mathbf{N}, \pi_n(x) \in \mathsf{F}(A^\bullet)\}$$

which is (in general) strictly included in $\mathsf{F}(A^\omega)$.

This problem is very reminiscent of the one in [14, section 5.2], where its causes, and a workaround, are detailed.